

**FRACTIONAL FOURIER TRANSFORM
AND SOME OF ITS APPLICATIONS**

Yury F. Luchko ¹, Héctor Matrínez ², Juan J. Trujillo ³

*Dedicated to Professor Anatoly Kilbas
on the occasion of his 60th anniversary*

Abstract

In the paper, a new definition of the fractional Fourier transform of the real order α is introduced. This transform plays the same role for the fractional derivatives as the Fourier transform for the ordinary derivatives does. If $\alpha = 1$, the fractional Fourier transform is reduced to the Fourier transform in the usual sense. Some important properties of the fractional Fourier transform including the inversion formula and the operational relations for the fractional derivatives are presented. Applications of the introduced transform for solving some model partial differential equations of fractional order are given.

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1. Introduction

Fractional Fourier transform in the form of fractional powers of the Fourier operator has been introduced as early as 1929 (see e.g. [16]). Later on, this notion has been applied in particular in quantum mechanics, chemistry, optics, dynamical systems, stochastic processes, and signal processing.

We refer the reader to the well written book [8] for the detailed presentation of applications related to the fractional Fourier transform, as well as to its mathematical background, properties, and references to the original literature.

In [8], the authors present 6 different definitions of the fractional Fourier transform. All of them were suggested to be used in different contexts like the voice, images or signal processing and work well with the fractional models, but there is no direct connection between these definitions and the Fractional Calculus known. Still, the situation with the definition of the fractional Fourier transform is very similar to the one by the fractional derivatives: Whereas there exists a lot of different definitions of the fractional derivatives, the answer to the question what definition to use depends mainly on the problem we are dealing with. There is no one best definition of the fractional derivative or the fractional Fourier transform – one should rather try to take the suitable one while modeling a process or considering a mathematical problem.

Our intention in this paper is to introduce a new definition of the fractional Fourier transform that is suitable to use while dealing with the fractional differential equations. It is well known that - under certain conditions - the following operational relation holds true (see e.g. the book [14]):

$$(\mathcal{F} D_{\pm}^{\alpha} u)(\omega) = (\mp i \omega)^{\alpha} (\mathcal{F} u)(\omega), \quad \alpha \geq 0, \quad (1)$$

where \mathcal{F} is the Fourier transform (2) and D_{\pm}^{α} are the suitable Riemann-Liouville fractional derivatives. This formula can be even interpreted as a definition of a fractional derivative. In fact, this and similar definitions are currently actively used by many researchers who employ Fractional Calculus to model applied problems in different fields to represent the spatial fractional derivatives. In particular, for a sufficiently well-behaved function u the *Riesz-Feller* space-fractional derivative of order α and skewness θ is defined as (see e.g. [7])

$$(\mathcal{F} D_{\theta}^{\alpha} u)(\omega) = -\psi_{\alpha}^{\theta}(\omega) (\mathcal{F} u)(\omega),$$

$$\psi_{\alpha}^{\theta}(\omega) = |\omega|^{\alpha} e^{i(\text{sign}(\omega))\theta\pi/2}, \quad 0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}.$$

Whereas the formula (1) can be used without any problems for the conventional derivatives ($\alpha = n \in \mathbb{N}$), in the case of an arbitrary positive α a certain branch of the multi-valued complex function $(\mp i \omega)^{\alpha}$ has to be chosen that might cause problems while applying this formula to several fractional derivatives with the different exponents and, in particular, to

the composition of such derivatives. Moreover, replacing a multi-valued complex function with one of its branches means information loss, so that the inverse of such function will be not identical to the original one. To avoid these problems, our motivation behind the suggested new definition was to introduce a fractionalization of the Fourier transform \mathcal{F}_α of order α that acts on the fractional derivative D^α of order α in the exact the same way as the conventional Fourier transform acts on the usual derivative, i.e.,

$$(\mathcal{F}_\alpha D^\alpha u)(\omega) = (-i c_\alpha \omega)(\mathcal{F}_\alpha u)(\omega),$$

with a constant c_α depending on the order α of the fractional derivative. It turns out, that our definition keeps most of the properties known in the literature for other definitions of the fractional Fourier transform. Moreover, handling of the so defined fractional Fourier transform mainly follows the lines of using the conventional Fourier transform and so can be managed by everybody familiar with it.

The remainder of the paper is organized as follows. The second section is devoted to the definition of the fractional Fourier transform and investigation of its properties including the inversion formula. In the third section we deal with the operational relations for the fractional Fourier transform and fractional derivatives. The last section contains some examples of application of the fractional Fourier transforms to the model partial differential equations of fractional order.

2. Fractional Fourier transform

For a function $u \in S$, S being the space of rapidly decreasing test functions on the real axis \mathbb{R} , the Fourier transform \hat{u} is defined as

$$\hat{u}(\omega) = (\mathcal{F}u)(\omega) = \int_{-\infty}^{+\infty} u(t) e^{i\omega t} dt, \quad \omega \in \mathbb{R}. \quad (2)$$

The inverse Fourier transform has the form

$$u(t) = (\mathcal{F}^{-1}\hat{u})(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\omega) e^{-i\omega t} d\omega, \quad t \in \mathbb{R}. \quad (3)$$

The operator \mathcal{F} can be extended to the space S' of tempered distributions (for definition of S' see e.g. [3]) following the standard procedure. For the theory of the Fourier transform in other spaces of functions see e.g. [15].

Now we introduce the space of the test functions where all formulae presented in the paper are valid. Of course, our results can be extended to the dual space of the generalized functions, to the L_p spaces or to the Sobolev

spaces; this will be done elsewhere. In this paper, we restrict ourselves to the Lizorkin space of the test functions that is defined as follows: Let S be the space of rapidly decreasing test functions. Denote by $V(\mathbb{R})$ the set of functions $v \in S$ satisfying the conditions:

$$\frac{d^n v}{dx^n} \Big|_{x=0} = 0, \quad n = 0, 1, 2, \dots$$

The Lizorkin space $\Phi(\mathbb{R})$ is introduced as the Fourier pre-image of the space $V(\mathbb{R})$ in the space S , i.e.,

$$\Phi(\mathbb{R}) = \{\varphi \in S : \hat{\varphi} \in V(\mathbb{R})\}.$$

According to the definition of the Lizorkin space, any function $\varphi \in \Phi(\mathbb{R})$ satisfies the orthogonality conditions

$$\int_{-\infty}^{+\infty} x^n \varphi(x) dx = 0, \quad n = 0, 1, 2, \dots$$

We note that the Lizorkin space and its dual space have been studied by several authors including [5], [6], [12], [14] [13], [11]. In particular, it was shown there that the Lizorkin space is invariant with respect to the fractional integration and differentiation operators (this is not the case for the whole space S of the rapidly decreasing test functions because the fractional integrals and derivatives of the functions from the space S not always belong to the space S). The reason mentioned above makes the Lizorkin space to be a very convenient one while dealing both with the Fourier transform and with the fractional integration and differentiation operators.

DEFINITION 2.1. For a function $u \in \Phi(\mathbb{R})$, the fractional Fourier transform of the order α ($0 < \alpha \leq 1$), \hat{u}_α , is defined as

$$\hat{u}_\alpha(\omega) = (\mathcal{F}_\alpha u)(\omega) = \int_{-\infty}^{+\infty} u(t) e_\alpha(\omega, t) dt, \quad \omega \in \mathbb{R}, \quad (4)$$

where

$$e_\alpha(\omega, t) := \begin{cases} e^{-i|\omega|^{1/\alpha}t}, & \omega \leq 0, \\ e^{i|\omega|^{1/\alpha}t}, & \omega \geq 0. \end{cases} \quad (5)$$

If $\alpha = 1$, the kernel e_α defined by (5) coincides with the kernel of the conventional Fourier transform:

$$e_1(\omega, t) := \begin{cases} e^{-i|\omega|t}, & \omega \leq 0 \\ e^{i|\omega|t}, & \omega \geq 0 \end{cases} \equiv e^{i\omega t}, \quad \omega \in \mathbb{R}, \quad t \in \mathbb{R}.$$

This means that the fractional Fourier transform of the order 1 is just the conventional Fourier transform: $\mathcal{F}_1 \equiv \mathcal{F}$.

The relation between the fractional and conventional Fourier transform is given by the following simple formula:

$$\hat{u}_\alpha(\omega) = (\mathcal{F}_\alpha u)(\omega) \equiv (\mathcal{F}u)(x) = \hat{u}(x), \quad (6)$$

where

$$x := \begin{cases} -|\omega|^{1/\alpha}, & \omega \leq 0, \\ |\omega|^{1/\alpha}, & \omega \geq 0. \end{cases} \quad (7)$$

Using the formulae (6)-(7) we can use the known properties of the Fourier transform both to calculate the fractional Fourier transform of the concrete functions and to determine the inversion of the fractional Fourier transform.

EXAMPLE 2.1. Let us evaluate the fractional Fourier transform of the function

$$u(t) := \begin{cases} A, & |t| \leq T, \\ 0, & |t| > T. \end{cases}$$

Using the well known formula for the Fourier transform of the function u and the relation (6), we get

$$\begin{aligned} (\mathcal{F}_\alpha u)(\omega) &= (\mathcal{F}u)(x) = \frac{A \sin(Tx)}{\pi x} \\ &= \begin{cases} \frac{A \sin(-T|\omega|^{1/\alpha})}{-\pi|\omega|^{1/\alpha}}, & \omega \leq 0, \\ \frac{A \sin(T|\omega|^{1/\alpha})}{\pi|\omega|^{1/\alpha}}, & \omega \geq 0. \end{cases} = \frac{A \sin(T|\omega|^{1/\alpha})}{\pi|\omega|^{1/\alpha}}. \end{aligned}$$

EXAMPLE 2.2. Let

$$(\mathcal{F}_\alpha u)(\omega) = \hat{u}_\alpha(\omega) = g(\omega).$$

Then

$$(\mathcal{F}_\alpha u)(\omega) = (\mathcal{F}u)(x) = g_1(x), \quad x := \begin{cases} -|\omega|^{1/\alpha}, & \omega \leq 0, \\ |\omega|^{1/\alpha}, & \omega \geq 0 \end{cases}$$

and

$$u(t) := (\mathcal{F}_\alpha^{-1} \hat{u}_\alpha)(t) = (\mathcal{F}^{-1} g_1)(t). \quad (8)$$

3. Operational relations for the fractional Fourier transform

In this section, we turn our attention to the operational relations between the fractional Fourier transform and the fractional derivatives that are defined in the paper as follows:

$$(D_\beta^\alpha u)(x) := (1 - \beta)(D_+^\alpha u)(x) - \beta(D_-^\alpha u)(x), \quad 0 < \alpha \leq 1, \quad \beta \in \mathbb{R}, \quad (9)$$

D_+^α and D_-^α being the Riemann-Liouville fractional derivatives on the real axis:

$$(D_+^\alpha u)(x) := \left(\frac{d}{dx} \right) (I_+^{1-\alpha} u)(x), \quad (10)$$

where I_+^α is the Riemann-Liouville fractional integral operator

$$(I_+^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} u(t) dt, \quad (11)$$

and

$$(D_-^\alpha u)(x) := \left(-\frac{d}{dx} \right) (I_-^{1-\alpha} u)(x), \quad (12)$$

where I_-^α is the Riemann-Liouville fractional integral operator

$$(I_-^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (t-x)^{\alpha-1} u(t) dt. \quad (13)$$

Let us remark that the fractional derivative D_β^α coincides with the ordinary derivative for any value of β if $\alpha = 1$:

$$(D_\beta^1 u)(x) = (1-\beta)(D_+^1 u)(x) - \beta(D_-^1 u)(x) = (1-\beta)\frac{du}{dx} + \beta\frac{du}{dx} = \frac{du}{dx}.$$

The most interesting particular cases of the fractional derivative (9) are the following ones:

- 1) $\beta = 0$: $D_0^\alpha \equiv D_+^\alpha$,
- 2) $\beta = 1$: $D_1^\alpha \equiv -D_-^\alpha$,
- 3) $\beta = 1/2$: $D_{1/2}^\alpha \equiv \frac{1}{2}(D_+^\alpha - D_-^\alpha)$.

The operator $D_{1/2}^\alpha$ can be interpreted as the one-dimensional inversion of the fractional Riesz potential and it is thus an object important for applications.

For the functions u, v from the Lizorkin space $\Phi(\mathbb{R})$ of functions the formula for integration by parts for the fractional Riemann-Liouville derivatives holds true (see [14] for the proof in the space L_p ; $\Phi(\mathbb{R}) \subset L_p$):

$$\int_{-\infty}^{+\infty} v(x)(D_+^\alpha u)(x) dx = \int_{-\infty}^{+\infty} (D_-^\alpha v)(x) u(x) dx. \quad (14)$$

Another result we need for the further discussions is connected with the fractional derivatives of the exponential functions $e^{i\omega t}$, $\omega \in \mathbb{R}$, $\omega \neq 0$.

LEMMA 3.1. *Let $\omega \in \mathbb{R}$, $\omega \neq 0$ and $0 < \alpha < 1$. Then*

$$(I_+^\alpha e^{i\omega t})(x) = e^{i\omega x} |\omega|^{-\alpha} (\cos(\alpha\pi/2) - i \operatorname{sign}(\omega) \sin(\alpha\pi/2)). \quad (15)$$

P r o o f. Using the variables substitution $\tau = x - t$ the integral $(I_+^\alpha e^{i\omega t})(x)$ can be represented in the form

$$\begin{aligned} (I_+^\alpha e^{i\omega t})(x) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} e^{i\omega t} dt e^{i\omega x} \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \tau^{\alpha-1} e^{-i\omega \tau} d\tau \\ &= e^{i\omega x} \left(\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \tau^{\alpha-1} \cos(\omega \tau) d\tau - i \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \tau^{\alpha-1} \sin(\omega \tau) d\tau \right). \end{aligned}$$

Both integrals in the last expression are convergent under the condition $\omega \in \mathbb{R}$, $\omega \neq 0$ and $0 < \alpha < 1$ and can be evaluated in the analytical form using the integral tables [9]:

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \tau^{\alpha-1} \cos(\omega \tau) d\tau &= |\omega|^{-\alpha} \cos(\alpha\pi/2), \\ \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \tau^{\alpha-1} \sin(\omega \tau) d\tau &= \operatorname{sign}(\omega) |\omega|^{-\alpha} \sin(\alpha\pi/2). \end{aligned}$$

The formula (15) follows now from the last three formulae. ■

In the similar way we can prove the

LEMMA 3.2. *Let $\omega \in \mathbb{R}$, $\omega \neq 0$ and $0 < \alpha < 1$. Then*

$$(I_-^\alpha e^{i\omega t})(x) = e^{i\omega x} |\omega|^{-\alpha} (\cos(\alpha\pi/2) + i \operatorname{sign}(\omega) \sin(\alpha\pi/2)). \quad (16)$$

We can now evaluate the fractional derivatives of the exponential functions $e^{i\omega t}$, $\omega \in \mathbb{R}$, $\omega \neq 0$.

LEMMA 3.3. *Let $\omega \in \mathbb{R}$, $\omega \neq 0$ and $0 < \alpha < 1$. Then*

$$(D_+^\alpha e^{i\omega t})(x) = e^{i\omega x} |\omega|^\alpha (\cos(\alpha\pi/2) + i \operatorname{sign}(\omega) \sin(\alpha\pi/2)). \quad (17)$$

P r o o f. We use Lemma 3.1 and obtain

$$\begin{aligned} (D_+^\alpha e^{i\omega t})(x) &= \frac{d}{dx} (I_+^{1-\alpha} e^{i\omega t})(x) \\ &= \frac{d}{dx} (e^{i\omega x} |\omega|^{-(1-\alpha)} (\cos((1-\alpha)\pi/2) - i \operatorname{sign}(\omega) \sin((1-\alpha)\pi/2))) \\ &= e^{i\omega x} (i\omega) |\omega|^{-(1-\alpha)} (\cos((1-\alpha)\pi/2) - i \operatorname{sign}(\omega) \sin((1-\alpha)\pi/2)) \\ &= e^{i\omega x} \operatorname{sign}(\omega) |\omega| |\omega|^{-(1-\alpha)} i (\sin(\alpha\pi/2) - i \operatorname{sign}(\omega) \cos(\alpha\pi/2)) \\ &= e^{i\omega x} |\omega|^\alpha (\cos(\alpha\pi/2) + i \operatorname{sign}(\omega) \sin(\alpha\pi/2)). \end{aligned}$$

■

In the similar way we can prove the

LEMMA 3.4. *Let $\omega \in \mathbb{R}$, $\omega \neq 0$ and $0 < \alpha < 1$. Then*

$$(D_-^\alpha e^{i\omega t})(x) = e^{i\omega x} |\omega|^\alpha (\cos(\alpha\pi/2) - i \operatorname{sign}(\omega) \sin(\alpha\pi/2)). \quad (18)$$

REMARK 3.1. The formulae (15)-(18) can be considered to be an extension of the well known formulae (see e.g. [4])

$$(I_+^\alpha e^{\lambda t})(x) = \lambda^{-\alpha} e^{\lambda x}, \quad \Re(\lambda) > 0, \quad \alpha > 0,$$

$$(I_-^\alpha e^{-\lambda t})(x) = \lambda^{-\alpha} e^{-\lambda x}, \quad \Re(\lambda) > 0, \quad \alpha > 0,$$

$$(D_+^\alpha e^{\lambda t})(x) = \lambda^\alpha e^{\lambda x}, \quad \Re(\lambda) > 0, \quad \alpha > 0,$$

$$(D_-^\alpha e^{-\lambda t})(x) = \lambda^\alpha e^{-\lambda x}, \quad \Re(\lambda) > 0, \quad \alpha > 0$$

to the case of $\Re(\lambda) = 0$, $\Im(\lambda) \neq 0$, $0 < \alpha < 1$.

We are now in a position to formulate and prove the main result of this section.

THEOREM 4.1. *Let $0 < \alpha \leq 1$ and a function u belong to the Lizorkin space $\Phi(\mathbb{R})$. Then the following operational relation holds true for any value of the parameter β :*

$$(\mathcal{F}_\alpha D_\beta^\alpha u)(\omega) = (-i c_\alpha \omega)(\mathcal{F}_\alpha u)(\omega), \quad \omega \in \mathbb{R}, \quad (19)$$

where c_α is a constant defined as

$$c_\alpha = \sin(\alpha\pi/2) + i \operatorname{sign}(\omega)(1 - 2\beta) \cos(\alpha\pi/2). \quad (20)$$

In particular, for the fractional derivative $D_{1/2}^\alpha = 1/2(D_+^\alpha - D_-^\alpha)$ the operational relation (19) can be represented in the standard form

$$(\mathcal{F}_\alpha D_{1/2}^\alpha u)(\omega) = (-i \sin(\alpha\pi/2) \omega)(\mathcal{F}_\alpha u)(\omega), \quad \omega \in \mathbb{R}. \quad (21)$$

REMARK 3.2. If $\alpha = 1$, the operational relation (19) is reduced to the known operational relation for the conventional Fourier transform for any value of the parameter β :

$$(\mathcal{F}_1 D_\beta^1 u)(\omega) = (\mathcal{F} \frac{d}{dx} u)(\omega) = (-i\omega)(\mathcal{F} u)(\omega), \quad \omega \in \mathbb{R}. \quad (22)$$

PROOF OF THE THEOREM 4.1: We separately consider several cases: 1) $\alpha = 1$, 2) $\alpha \neq 1$, $\omega = 0$, 3) $\alpha \neq 1$, $\omega > 0$, 4) $\alpha \neq 1$, $\omega < 0$.

Case 1: If $\alpha = 1$, the statement of the Theorem 4.1 is just the classical result for the conventional Fourier transform (see Remark 3.2).

Case 2: If $\omega = 0$, we have to show that

$$(\mathcal{F}_\alpha D_\beta^\alpha u)(0) = 0 \quad (23)$$

for any function u from the Lizorkin space $\Phi(\mathbb{R})$. Indeed,

$$\begin{aligned} (\mathcal{F}_\alpha D_\beta^\alpha u)(0) &= \int_{-\infty}^{+\infty} (D_\beta^\alpha u)(x) dx \\ &= \int_{-\infty}^{+\infty} \left[(1-\beta) \frac{d}{dx} (I_+^{1-\alpha} u)(x) + \beta \frac{d}{dx} (I_-^{1-\alpha} u)(x) \right] dx \\ &= [(1-\beta)(I_+^{1-\alpha} u)(x) + \beta(I_-^{1-\alpha} u)(x)] \Big|_{-\infty}^{+\infty}. \end{aligned}$$

The last expression is equal to 0 because the fractional integrals of a function u from the Lizorkin space $\Phi(\mathbb{R})$ belongs to the space $\Phi(\mathbb{R})$, too (see [14]). The inclusion $\Phi(\mathbb{R}) \subset S$, S being the space of rapidly decreasing test functions on the real axis \mathbb{R} , finishes the proof in this case.

Case 3: We use now the formula (14) for integration by parts for the functions $v = e^{i|\omega|^{1/\alpha}x}$ and $u \in \Phi(\mathbb{R})$. Of course, the function v does not belong to the Lizorkin space $\Phi(\mathbb{R})$ or even to L_p and thus we cannot apply the formula immediately. But v is a bounded function with respect to the variables x and ω : $|v| = 1 \ \forall x, \omega \in \mathbb{R}$ and thus the formula (14) can be directly proved by means of the Fubini theorem for any function $u \in \Phi(\mathbb{R})$. Using the formulae (14), (17), (18) we have now the chain of identities ($\omega > 0$):

$$\begin{aligned} (\mathcal{F}_\alpha D_\beta^\alpha u)(\omega) &= \int_{-\infty}^{+\infty} e^{i|\omega|^{1/\alpha}x} (D_\beta^\alpha u)(x) dx \\ &= (1-\beta) \int_{-\infty}^{+\infty} e^{i|\omega|^{1/\alpha}x} (D_+^\alpha u)(x) dx - \beta \int_{-\infty}^{+\infty} e^{i|\omega|^{1/\alpha}x} (D_-^\alpha u)(x) dx \\ &= (1-\beta) \int_{-\infty}^{+\infty} (D_-^\alpha e^{i|\omega|^{1/\alpha}x})(x) u(x) dx - \beta \int_{-\infty}^{+\infty} (D_+^\alpha e^{i|\omega|^{1/\alpha}x})(x) u(x) dx \\ &= (1-\beta) \int_{-\infty}^{+\infty} e^{i|\omega|^{1/\alpha}x} |\omega| (\cos(\alpha\pi/2) - i \sin(\alpha\pi/2)) u(x) dx \\ &\quad - \beta \int_{-\infty}^{+\infty} e^{i|\omega|^{1/\alpha}x} |\omega| (\cos(\alpha\pi/2) + i \sin(\alpha\pi/2)) u(x) dx \\ &= ((1-2\beta) \cos(\alpha\pi/2) - i \sin(\alpha\pi/2)) \omega \int_{-\infty}^{+\infty} e^{i|\omega|^{1/\alpha}x} u(x) dx \\ &= (-i\omega) (\sin(\alpha\pi/2) + i(1-2\beta) \cos(\alpha\pi/2)) (\mathcal{F}_\alpha u)(\omega). \end{aligned}$$

Case 4 ($\alpha \neq 1, \omega < 0$) is like Case 3, if we take the function $v = e^{-i|\omega|^{1/\alpha}x}$ instead of $v = e^{i|\omega|^{1/\alpha}x}$, so we omit the intermediate calculations:

$$\begin{aligned} (\mathcal{F}_\alpha D_\beta^\alpha u)(\omega) &= \int_{-\infty}^{+\infty} e^{-i|\omega|^{1/\alpha}x} (D_\beta^\alpha u)(x) dx \\ &= (-i\omega)(\sin(\alpha\pi/2) - i(1 - 2\beta)\cos(\alpha\pi/2))(\mathcal{F}_\alpha u)(\omega). \end{aligned}$$

Putting Cases 1)-4) together completes the proof of the operational relation (19). Finally, the operational relation (19) results in the formula (21), if $\beta = 1/2$. ■

REMARK 3.3. The case $\beta = 1/2$ is the most interesting one (see the operational relation (21)). It can be considered to be a direct fractional generalization of the operational relation (22) for the conventional Fourier transform. Other potentially useful cases are as follows:

1) $\beta = 0, D_0^\alpha \equiv D_+^\alpha$:

$$\begin{aligned} (\mathcal{F}_\alpha D_+^\alpha u)(\omega) &= (-i c_\alpha \omega)(\mathcal{F}_\alpha u)(\omega), \quad \omega \in \mathbb{R}, \\ c_\alpha &= (\sin(\alpha\pi/2) + i \operatorname{sign}(\omega) \cos(\alpha\pi/2)). \end{aligned}$$

2) $\beta = 1, D_0^\alpha \equiv -D_-^\alpha$:

$$\begin{aligned} (\mathcal{F}_\alpha D_-^\alpha u)(\omega) &= (-i c_\alpha \omega)(\mathcal{F}_\alpha u)(\omega), \quad \omega \in \mathbb{R}, \\ c_\alpha &= (\sin(\alpha\pi/2) - i \operatorname{sign}(\omega) \cos(\alpha\pi/2)). \end{aligned}$$

REMARK 3.4. Of course, the results presented in this section, especially the ones stated in Theorem 4.1, can be proved in other classical spaces of functions following the standard procedure.

4. Applications of the fractional Fourier transform

In this section the way of how to use the operational relation (21) for solving the linear partial differential equations of fractional order is indicated.

We consider the space-time fractional diffusion equation

$${}_x D_{1/2}^\alpha u(x, t) = {}_t D_*^\beta u(x, t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad (24)$$

where the α, β are real parameters always restricted as follows $0 < \alpha \leq 1, 0 < \beta \leq 2, {}_x D_{1/2}^\alpha = \frac{1}{2}({}_x D_+^\alpha - {}_x D_-^\alpha)$ is the space-fractional derivative

of order α and ${}_tD_*^\beta$ is the Caputo time-fractional derivative of order β ($m-1 < \beta \leq m$, $m \in \mathbb{N}$) defined as follows:

$${}_tD_*^\beta f(t) := \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\beta+1-m}}, & m-1 < \beta < m, \\ \frac{d^m}{dt^m} f(t), & \beta = m. \end{cases} \quad (25)$$

The operator defined by (25) has been referred to as the *Caputo* fractional derivative since it was introduced by Caputo in the late 1960's for modeling the energy dissipation in some anelastic materials with memory, see [2].

It is well known that for a sufficiently well-behaved function f the property

$$\mathcal{L}\{{}_tD_*^\beta f(t); s\} = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k} f^{(k)}(0^+), \quad m-1 < \beta \leq m \quad (26)$$

holds true, \mathcal{L} being the Laplace transform

$$\tilde{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \quad \Re(s) > a_f$$

of a function f . A sufficient condition of the existence of the Laplace transform is that the original function is of exponential order as $t \rightarrow \infty$. This means that some constant a_f exists such that the product $e^{-a_f t} |f(t)|$ is bounded for all t greater than some T . Then $\tilde{f}(s)$ exists and is analytic in the half plane $\Re(s) > a_f$.

For the equation (24) we consider the Cauchy problem

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{R}, \quad u(\pm\infty, t) = 0, \quad t > 0, \quad (27)$$

where $\varphi(x) \in L^c(\mathbb{R})$ is a sufficiently well-behaved function. If $1 < \beta \leq 2$ we add the condition $u_t(x, 0) = 0$, where $u_t(x, t) = \frac{\partial}{\partial t} u(x, t)$.

By solution of the Cauchy problem for the equation (24) we mean a function u which satisfies the conditions (27). The Green function (or fundamental solution) of the Cauchy problem is a (generalized) function G that corresponds to the initial condition $\varphi(x) = \delta(x)$, δ being the Dirac delta function.

Taking into account the operational relation (21) for the space-fractional derivative and the Laplace transform formula (26) for the Caputo time-fractional derivative we get the following formula for the Laplace transform and the fractional Fourier transform of order α of the Green function:

$$-i \sin(\alpha\pi/2) \omega \widehat{\widehat{G_\alpha}}(\omega, s) = s^\beta \widehat{\widehat{G_\alpha}}(\omega, s) - s^{\beta-1}.$$

It follows from the last equation that

$$\widehat{\widehat{G}}_{\alpha}(\omega, s) = \frac{s^{\beta-1}}{s^{\beta} + i \sin(\alpha\pi/2)\omega}. \quad (28)$$

To find the Green function let us first invert the Laplace transform in the last formula. For this purpose we recall the well known Laplace transform pair (see e.g. [7] and references therein),

$$E_{\beta}(ct^{\beta}) \stackrel{\mathcal{L}}{\leftrightarrow} \frac{s^{\beta-1}}{s^{\beta} - c}, \quad \Re(s) > |c|^{1/\beta}, \quad (29)$$

with $c \in \mathcal{C}$, $0 < \beta \leq 2$, where E_{β} denotes the Mittag-Leffler function of order β , defined in the complex plane by the power series

$$E_{\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, \quad z \in \mathcal{C}.$$

Then, by comparing (28) with (29), we obtain the fractional Fourier transform of the Green function as

$$\widehat{\widehat{G}}_{\alpha}(\omega, t) = E_{\beta} \left[-i \sin(\alpha\pi) \omega t^{\beta} \right], \quad \kappa \in \mathbb{R}, \quad t \geq 0. \quad (30)$$

The Green function G of the initial-value problem (27) for the equation (24) can be expressed in the form

$$G(x, t) = (\mathcal{F}_{\alpha}^{-1} E_{\beta} \left[-i \sin(\alpha\pi) \omega t^{\beta} \right])(x), \quad (31)$$

$\mathcal{F}_{\alpha}^{-1}$ being the inverse fractional Fourier transform given by (8). Making use of the Mellin-Barnes representation (see e.g. [1], [10]) of the Mittag-Leffler function E_{β} in the form

$$E_{\beta}(z) = \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\pi (-z)^s ds}{\Gamma(1 + \beta s) \sin s\pi} = \frac{1}{2\pi i} \int_{L_{-\infty}} \frac{\Gamma(s) \Gamma(1-s)}{\Gamma(1 - \beta s)} (-z)^{-s} ds,$$

where the integration is over a left-hand loop $L_{-\infty}$ drawn round all the left-hand poles $s = 0, -1, -2, \dots$ of the integrand in a positive direction and the Mellin transform machinery, we can invert the fractional Fourier transform (31) and represent the Green function G in the form of a linear combination of certain Mellin-Barnes integrals. The technique used to do this is indeed the same one that has been employed in the paper [7] for the case of the space-time fractional diffusion equation with the Riesz-Feller fractional derivative instead of the fractional derivative ${}_x D_{1/2}^{\alpha}$. These results in form of the generalized hyper-geometric series as well as asymptotics and numerical evaluations of the Green function for the equation (24) will be presented elsewhere.

REMARK 4.1. Like in the case of the conventional Fourier transform, the discrete fractional Fourier and the fast fractional Fourier transforms can be introduced, studied, and applied to various problems in different fields. All these notions are the topics for further research.

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¹ *Dept. of Mathematics II*
Technical University of Applied Sciences Berlin
Luxemburger Str. 10 , 13353 Berlin – GERMANY
e-mail: luchko@tfh-berlin.de

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² *National Experimental Univ. of Guayana*
Port Ordaz, State of Bolivar, VENEZUELA

³ *Dept. de Analisis Matematico*
Universidad de La Laguna
38 271 – La Laguna (Tenerife), Canary Islands – SPAIN
e-mail: jtrujill@ullmat.es